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THEORY OF ELASTIC-PLASTIC DEFORMATION OF RANDOMLY REINFORCED COMPOSITE MATERIALS

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Using the methods of the mechanics of random inhomogeneous media, we study the elastic-plastic properties of a composite material containing randomly oriented ellipsoidal inclusions. The analogous problem for composites with spherical inclusions and matrix mixtures was solved in [1].

1. Let a composite material occupying volume V and bounded by surface S be formed of an elastic-plastic matrix and randomly oriented ellipsoidal inclusions of identical form. The governing equations for the materials of both components, bonded together with ideal adhesion, are given by

$$s_{ij} = 2\mu_m(e)e_{ij},\ \sigma_{pp} = 3K_m\varepsilon_{pp},\ s_{ij} = 2\mu_f(e)e_{ij},\ \sigma_{pp} = 3K_f\varepsilon_{pp}. \tag{1.1}$$

Here $s_{ij} = \sigma_{ij} - (1/3)\delta_{ij}\sigma_{pp}$; $e_{ij} = \varepsilon_{ij} - (1/3)\delta_{ij}\varepsilon_{pp}$; σ_{ij}, ε_{ij} are the components of the stress and deformation tensors; $\mu_{m,f}(e)$ are the plastic shear moduli; $K_{m,f}$ are the bulk moduli of the material components ($K_{m,f}$ = const); $e = \sqrt{e_{ij}e_{ij}}$. The index m refers to the matrix material; f to that of the inclusions.

We will describe the structure of the composite by using the indicator function $\kappa(\mathbf{r})$, which is equal to zero in the matrix volume V_m and to unity in the inclusion volume V_f. In addition, the spatial position of the ellipsoids is given by a collection of indicator functions $\varkappa_1(\mathbf{r}), \varkappa_2(\mathbf{r}), \ldots, \varkappa_n(\mathbf{r})$. Each function $\kappa_s(\mathbf{r})$ is equal to unity in the volume V_s of all ellipsoids oriented in direction s and is equal to zero outside of this volume. Clearly $\varkappa(\mathbf{r}) = \sum_{s=1}^{n} \varkappa_s(\mathbf{r})$. By using $\kappa(\mathbf{r})$, (1.1) can be written in the form

$$\begin{aligned} s_{ij}(\mathbf{r}) &= 2(\mu_m(e) + (\mu_f(e) - \mu_m(e))\varkappa(\mathbf{r}))e_{ij}(\mathbf{r}),\\ \sigma_{pp}(\mathbf{r}) &= 3(K_m + (K_f - K_m)\varkappa(\mathbf{r}))\varepsilon_{pp}(\mathbf{r}). \end{aligned} \tag{1.2}$$

All of the indicator functions, stresses and deformations are assumed to be statistically uniform and ergodic random fields, and their expectation values are replaced by volume-averaged values [2]:

$$\langle f\rangle = \frac{1}{V}\int_V f(\mathbf{r})\,d\mathbf{r},\quad \langle f\rangle_{m,f,s} = \frac{1}{V_{m,f,s}}\int_{V_{m,f,s}} f(\mathbf{r})\,d\mathbf{r}\ (s = 1, 2, \ldots, n).$$

To find the macroscopic governing equations and the effective characteristics of these composites it is necessary to establish a connection between the macroscopic quantities $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$:

$$\langle\sigma_{ij}\rangle = E^*_{ijkl}\langle\varepsilon_{kl}\rangle, \tag{1.3}$$

where $E_{ijk\ell}^*$ are the components of the plastic moduli tensor, a function of the numerical characteristics of the random deformation field $\varepsilon_{ij}(\mathbf{r})$. Here and below an asterisk denotes the root mean square of the quantity.

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Relations (1.3) are obtained by statistically averaging the system of equations for elastic-plastic deformation of a composite material. This system consists of (1.2), the equilibrium equation

$$\sigma_{ip,p}(\mathbf{r}) = 0 \tag{1.4}$$

and the Cauchy formula

$$2\varepsilon_{ij}(\mathbf{r}) = u_{i,j}(\mathbf{r}) + u_{j,i}(\mathbf{r}), \tag{1.5}$$

expressing the components of the tensor of small elastic-plastic deformations through the components of the displacement vector $u_i(\mathbf{r})$. The boundary conditions for system (1.2), (1.4), and (1.5) are the conditions of homogeneity of the fluctuating quantities on the surface S of the volume V: $f(\mathbf{r}) = \langle f \rangle$, $\mathbf{r} \in S$.

In order to use the methods of the theory of elasticity for microscopically inhomogeneous media to establish effective relations, it is necessary to linearize (1.2). As in [1], we replace, under the plastic modulus sign, each component of the deformation in the limit by its expectation value:

$$\mu_{m,f}(e) = \mu_{m,f}(e_{m,f}), \quad e_{m,f} = \sqrt{\langle e_{ij}\rangle_{m,f}\langle e_{ij}\rangle_{m,f}}.$$

Relations (1.2) take the form

$$s_{ij}(\mathbf{r}) = 2(\mu_m + [\mu]\varkappa(\mathbf{r}))e_{ij}(\mathbf{r}),\ \sigma_{pp}(\mathbf{r}) = 3(K_m + [K]\varkappa(\mathbf{r}))\varepsilon_{pp}(\mathbf{r}). \tag{1.6}$$

Here $[\mu] = \mu_f(e_f) - \mu_m(e_m)$; $[K] = K_f - K_m$. We note that formulas (1.6) which are linear with respect to the local fields $\sigma_{ij}(\mathbf{r})$, $\varepsilon_{ij}(\mathbf{r})$, remain physically nonlinear, since the numerical characteristic of the random field $\varepsilon_{ij}(\mathbf{r})$ enters in a random fashion into the right-hand sides of these relations.

With the help of Green's tensor

$$G_{ik}(\mathbf{r}) = \frac{1}{8\pi\mu_m(e_m)}\left(\delta_{ik}r_{,pp} - \frac{3K_m + \mu_m(e_m)}{3K_m + 4\mu_m(e_m)}r_{,ik}\right), \quad r = |\mathbf{r}|$$

Eqs. (1.4)-(1.6) reduce to a system of integral equations

$$\varepsilon'_{ij}(\mathbf{r}) = \int\limits_V G_{ik,lj}(\mathbf{r} - \mathbf{r}_1)\tau'_{kl}(\mathbf{r}_1)\,d\mathbf{r}_1, \tag{1.7}$$

where $\tau_{kl}(\mathbf{r}) = -(2[\mu]e_{kl}(\mathbf{r}) + \delta_{kl}[K]\varepsilon_{pp}(\mathbf{r}))\varkappa(\mathbf{r})$; the primes denote fluctuations of the quantities in the volume V.

We now calculate the macroscopic stress $\langle\sigma_{ij}\rangle$. To do this, we average (1.6) over V and apply the mechanical mixing rule to the volume of the ellipsoids V_s:

$$\langle s_{ij}\rangle = 2\mu_m\langle e_{ij}\rangle + 2[\mu]\sum_{s=1}^{n} c_s\langle e_{ij}\rangle_s,$$

$$\langle\sigma_{pp}\rangle = 3K_m\langle\varepsilon_{pp}\rangle + 3[K]\sum_{s=1}^{n} c_s\langle\varepsilon_{pp}\rangle_s \tag{1.8}$$

($c_s = V_sV^{-1}$ is the volume content of ellipsoids of direction s). Formulae (1.8) show that to establish macroscopic governing equations for the composite, the averages over V_s of the deformations $\langle\varepsilon_{ij}\rangle_s$ are found from the well-known relations [1]

$$\langle\varepsilon_{ij}\rangle_s = \langle\varepsilon_{ij}\rangle + c_s^{-1}\langle\varkappa'_s\varepsilon'_{ij}\rangle \tag{1.9}$$

after determination of $\langle\varkappa'_s\varepsilon'_{ij}\rangle$. We compute this value with the help of (1.7). Multiplying by $\varkappa'_s(\mathbf{r})$ and averaging over V, we obtain

$$\langle\varkappa'_s\varepsilon'_{ij}\rangle = \int\limits_V G_{ik,lj}(\mathbf{r}_1)\langle\varkappa'_s(\mathbf{r})\tau'_{kl}(\mathbf{r} + \mathbf{r}_1)\rangle\,d\mathbf{r}_1.$$

To compute this integral, we use the fact that the function $\varkappa'_s(\mathbf{r})$ describes only ellipsoids of one orientation, and we assume that the correlation function in the integrand has the form

$$\langle \varkappa_s'(\mathbf{r}_1)\,\tau_{kl}'(\mathbf{r}+\mathbf{r}_1)\rangle = f_{kl}^{(s)}\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\right)$$

(a_i are the semiaxes of the ellipsoids). Such an assumption is a generalization of the strong isotropy hypothesis for the case of ellipsoidal anisotropies of one orientation [2, 3]. Using these assumptions, the value of the integral is expressed by

$$\langle \varkappa_s'\varepsilon_{ij}'\rangle = \frac{c_s}{2\mu_m} Z_{ijkl}^{(s)}\left(\langle\tau_{kl}\rangle_s - c_f\langle\tau_{kl}\rangle_f\right),$$

$$Z_{ijkl}^{(s)} = S_{ijkl}^{(s)} - \frac{\nu_m}{1+\nu_m} S_{ppkl}^{(s)}\delta_{ij}, \quad \nu_m = \frac{3K_m - 2\mu_m}{6K_m + 2\mu_m}, \tag{1.10}$$

where $c_f = V_f V^{-1}$ is the volume content of all inclusions. $S_{ijk\ell}{}^{(s)}$ are the components of the Eshelby tensor written in the laboratory frame of reference of the ellipsoids of direction s [4].

Substitution of the expressions for $\tau_{ij}(\mathbf{r})$ and (1.10) into (1.9) gives

$$\langle\varepsilon_{ij}\rangle_s = c_f Q_{ijpq}^{(s)} P_{pqkl}^{(s)}\langle\varepsilon_{kl}\rangle_f + Q_{ijkl}^{(s)}\langle\varepsilon_{kl}\rangle. \tag{1.11}$$

Here

$$P_{ijkl}^{(s)} = \frac{1}{2\mu_m}\left(2[\mu]\,Z_{ijkl}^{(s)} + \delta_{kl}[\lambda]\,Z_{ijpp}^{(s)}\right); \quad Q_{ijkl}^{(s)} =$$

$$= \left(I_{ijkl} + P_{ijkl}^{(s)}\right)^{-1}; \quad \lambda_{m,f} = K_{m,f} - \frac{2}{3}\mu_{m,f}.$$

By multiplying (1.11) by c_s and summing over all possible orientations of the ellipsoids, we can write the equations for the deformation rates averaged over V_f:

$$\langle\varepsilon_{ij}\rangle_f = \frac{\langle\varepsilon_{kl}\rangle - c_f\langle\varepsilon_{kl}\rangle_f}{c_f c_m}\sum_{s=1}^{n} c_s Q_{ijkl}^{(s)}. \tag{1.12}$$

Solving the tensor equations (1.12) for $\langle\varepsilon_{ij}\rangle_f$, we obtain

$$\langle\varepsilon_{ij}\rangle_f = a_{ijkl}\langle\varepsilon_{kl}\rangle, \tag{1.13}$$

where

$$a_{ijkl} = (I_{ijpq} + R_{ijpq})^{-1} R_{pqkl};$$

$$R_{ijkl} = \frac{1}{c_f c_m}\sum_{s=1}^{n} c_s Q_{ijkl}^{(s)}; \quad c_m = 1 - c_f.$$

Substituting (1.9) and (1.13) into (1.8), we find the macroscopic law of elastic-plastic deformation for the composite material in question:

$$\langle\sigma_{ij}\rangle = E_{ijkl}^{*}(e_{m,f})\,\langle\varepsilon_{kl}\rangle \tag{1.14}$$

($E_{ijk\ell}^{*} = 2\mu_m I_{ijk\ell} + \delta_{ij}\delta_{k\ell}\lambda_m + c_f(2[\mu]I_{ijpq} + \delta_{ij}\delta_{pq}[\lambda])a_{pqk\ell}$ is the effective plastic moduli tensor).

The system of equations for the macroscopic deformation of a composite (1.11), (1.13), and (1.14) contains, along with the macroscopic quantities $\langle\sigma_{ij}\rangle$, $\langle\varepsilon_{ij}\rangle$, the components of the deformation tensors $\langle\varepsilon_{ij}\rangle_{m,f}$. To compute these it is necessary to assign a form to the plastic modulus functions $\mu_{m,f}(e)$, which is determined according to the deformation properties of the component materials.

In the case where the $\mu_{m,f}$ are constants, (1.11), (1.13), and (1.14) coincide with the well-known results for elastic composites containing ellipsoidal inclusions [4].

2. An important special case of the general relations (1.14) is a composite material model in which the ellipsoidal inclusions are equiprobably oriented. The volume contents

of the ellipsoids of all directions in this composite are identical ($c_1 = c_2 = \ldots = c_n$), so that the fourth rank tensor $\sum_{s=1}^{n} c_s Q_{ijkl}^{(s)}$ is isotropic and can be represented in the form [5]

$$\sum_{s=1}^{n} c_s Q_{ijkl}^{(s)} = c_f (I_{ijkl}\alpha - \delta_{ij}\delta_{kl}\beta). \tag{2.1}$$

Here $\alpha = (3Q_{pqpq} - Q_{ppqq})/15$; $\beta = (Q_{pqpq} - 2Q_{ppqq})/15$ are the invariants of the tensor Q_{ijkl}. Substituting (2.1) into (1.13), (1.14) and separating the deviatoric and volume parts, we find the deformation averaged over the inclusions V_f:

$$\langle e_{ij}\rangle_f = \frac{\alpha}{c_m + \alpha c_f}\langle e_{ij}\rangle, \quad \langle \varepsilon_{pp}\rangle_f = \frac{\gamma}{c_m + \gamma c_f}\langle \varepsilon_{pp}\rangle. \tag{2.2}$$

Relation (1.14) takes the form

$$\langle s_{ij}\rangle = 2\mu^*(e_{m,f})\langle e_{ij}\rangle, \quad \langle \sigma_{pp}\rangle = 3K^*(e_{m,f})\langle \varepsilon_{pp}\rangle,$$
$$\mu^* = \mu_m + [\mu]\frac{c_f\alpha}{c_m + \alpha c_f}, \quad K^* = K_m + [K]\frac{c_f\gamma}{c_m + \gamma c_f}, \quad \gamma = \alpha - 3\beta \tag{2.3}$$

(μ^*, K^* are the effective plastic moduli of the isotropic composite). Relations (2.3) must be supplemented by equations for the relative values of $e_{m,f}$. From (2.2) and the mechanical mixing rule we have

$$e_m = \frac{1}{c_m + \alpha c_f}e, \quad e_f = \frac{\alpha}{c_m + \alpha c_f}e. \tag{2.4}$$

We apply (2.3) and (2.4) to the calculation of the elastic-plastic properties of composite samples prepared from sintered aluminum powders (SAP). Such composites, formed during sintering of aluminum powders, consists of an aluminum matrix in which particles of aluminum oxide Al_2O_3 are randomly distributed. These particles are plates of thickness h = 0.055 μm and linear dimension L = 10-16 μm [6]. We approximate the oxide particles by ellipsoids of revolution (oblate spheroids) with a semi-axis ratio of $\xi = hL^{-1}$. The components of the Eshelby tensor are expressed in this case by elementary functions, and the procedure for computing the invariants α and γ is easily carried out on a computer [7].

We construct a load–extension curve for this composite using (2.3) and (2.4). Equation (2.3) for uniaxial extension takes the form

$$\langle \sigma_{11}\rangle = \frac{9K^*\mu^*}{4K^* + \mu^*}\langle \varepsilon_{11}\rangle. \tag{2.5}$$

We will assign a form to the plastic moduli functions $\mu_{m,f}(e)$. In accordance with [1], parts of the load–extension curve for the aluminum matrix are approximated by an exponential dependence, for which

$$\mu_m(e) = \frac{k_m}{2e}\left(1 - \exp\left(-\frac{2G_m e}{k_m}\right)\right). \tag{2.6}$$

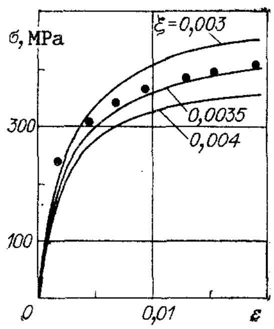

Fig. 1

Here G_m is the shear modulus; k_m is the limiting shear stress for this part of the curve (the flow limit). We will consider the inclusion material (high strength aluminum oxide particles with large moduli) as ideally elastic throughout the deformation process: μ_f = const. Equations (2.4)-(2.6) were solved numerically by computer using the method of successive approximations.

Figure 1 displays a comparison of theoretical and experimental load-extension curves for an SAP composite (14% Al_2O_3). The experimental results, taken from [8], are shown as points in Fig. 1. The computed values from formulas (2.4)-(2.6) are shown as solid lines. The calculated quantities are: E_m = 71 GPa; E_f = 2500 GPa; ν_m = 0.34; ν_f = 0.2; k_m = 25 MPa; c_f = 0.14.

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STABILITY OF A VISCOELASTIC ROD WITH A SPORADIC LONGITUDINAL LOAD

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The stability in an infinite time interval is studied for a viscoelastic rod compressed by a sporadic force. Rod bending is considered in a dynamic arrangement. Stability conditions are formulated in a root-mean-square for a viscoelastic rod with an arbitrary form of degree of stress relaxation and different types of end fastening. It is shown that with fulfillment of the conditions obtained a viscoelastic rod is stable, but a corresponding elastic rod with a long-term elasticity modulus is unstable. Questions of stability for a rod made of aging viscoelastic material with an arbitrary relaxation nucleus were considered in [1, 2]. The problem was studied in a quasistatic arrangement with a deterministic compressive load. A review of studies of the stability of viscoelastic structural elements is contained in [3]. Stability conditions for elastic bodies with a sporadic load are given in [4]. The stability elastic and viscoelastic rods with a sporadic longitudinal load is analyzed in [5-7]. Adequate stability conditions for viscoelastic rods are obtained in this work by means of the second Lyapunov method for a system with an aftereffect.

1. Model of a Viscoelastic Body. Before application of an external load the body is in a natural condition, and at instant of time t = 0 a force is applied to it under whose

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